Houston Journal of Mathematics
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Volume 34, No. 2, 2008

SOME FINITENESS CONDITIONS ON THE SET OF OVERRINGS OF A ϕ-RING

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Communicated by Klaus Kaiser

Abstract. Let $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$. For a ring $R \in \mathcal{H}$ with total quotient ring $T(R)$, let ϕ be the natural ring homomorphism from $T(R)$ into $R_{Nil(R)}$. An integral domain R is said to be an FC-domain (in the sense of Gilmer) if each chain of distinct overrings of R is finite, and R is called an FO-domain if R has finitely many overrings. A ring R is called an *FC-ring* if each chain of distinct overrings of R is finite, and R is said to be an *FO-ring* if R has finitely many overrings. A ring $R \in \mathcal{H}$ is said to be a ϕ-FC-ring if $\phi(R)$ is an FC-ring, and R is called a ϕ-FO-ring if $\phi(R)$ is an FO-ring. In this paper, we show that the theory of ϕ-FC-rings and ϕ-FO-rings resembles that of FC-domains and FO-domains.

1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, R' denotes the integral closure of R in $T(R)$, $Nil(R)$ denotes the set of nilpotent elements of R, $Z(R)$ denotes the set of zerodivisors of R. Recall from [19] and [9] that a prime ideal of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable (under set inclusion) to every ideal of R. Throughout this paper, $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$, and $\mathcal{H}_0 = \{R \in \mathcal{H} \mid Nil(R) = Z(R)\}$. In [7], [8], [10], [11], [12], and [13] the first-named author investigated the class of rings $\mathcal{H}$. Observe that if R is an integral domain, then $R \in \mathcal{H}_0 \subset \mathcal{H}$. If $R \in \mathcal{H}$, then R is called a ϕ-ring. For

2000 *Mathematics Subject Classification.* Primary 13B02, 13B22, and 13F05.
Key words and phrases. Divided prime, overring, finite chain of overrings, finite prime spectrum, Prüfer rings, Chained rings.

a further study on ϕ-rings, we recommend the references: [3], [4], [14], [15], and [16].

A non-zerodivisor of a ring R is called a *regular element* and an ideal of R is said to be *regular* if it contains a regular element. An ideal I of a ring R is said to be a *nonnil ideal* if $I \not\subseteq Nil(R)$. If I is a nonnil ideal of a ring $R \in \mathcal{H}$, then $Nil(R) \subset I$. In particular, $Nil(R) \subset I$ for every regular ideal of a ring $R \in \mathcal{H}$. Recall from [8] that for a ring $R \in \mathcal{H}$ with total quotient ring $T(R)$, the map $\phi : T(R) \longrightarrow R_{Nil(R)}$ such that $\phi(a/b) = a/b$ for $a \in R$ and $b \in R \setminus Z(R)$ is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{Nil(R)}$ given by $\phi(x) = x/1$ for every $x \in R$. Recall that if every finitely generated regular ideal of a ring R is invertible, then R is said to be a *Prüfer ring*. Recall from [3] that a nonnil ideal I of $R \in \mathcal{H}$ is a *ϕ-invertible* if $\phi(I)$ is an invertible ideal of $\phi(R)$, and a ring $R \in \mathcal{H}$ is said to be a *ϕ-Prüfer ring* if every finitely generated nonnil ideal of R is ϕ-invertible, that is, if $\phi(R)$ is a Prüfer ring. Also recall from [11] that a ring $R \in \mathcal{H}$ is said to be a ϕ-chained ring (ϕ-CR) if for each $x \in R_{Nil(R)} \setminus \phi(R)$, we have $x^{-1} \in \phi(R)$.

In this paper, we generalize the concept of FC-domains and FO-domains as in [22] to the context of rings that are in $\mathcal{H}$. Recall from [22] that an integral domain R is said to be an FC-domain if each chain of distinct overrings of R is finite, and R is called an FO-domain if R has finitely many overrings. Recall that B is said to be an overring of a ring R if $R \subseteq B \subseteq T(R)$, where $T(R)$ is the total quotient ring of R. Jaballah (the second-named author) asked in [27, Question 1] for a characterization of FO-domain. Gilmer in [22] gave such characterization. A ring R is called an *FC-ring* if each chain of distinct overrings of R is finite, and R is said to be an *FO-ring* if R has finitely many overrings. A ring $R \in \mathcal{H}$ is said to be a ϕ-FC-ring if $\phi(R)$ is an FC-ring, and R is called a ϕ-FO-ring if $\phi(R)$ is an FO-ring.

We remind the reader with the following important properties of ϕ-rings (for (1) through (5) see [8].) Let $R \in \mathcal{H}$. Then

(1) $\phi(R) \in \mathcal{H}_0$.
(2) $Ker(\phi) \subseteq Nil(R)$.
(3) $Nil(T(R)) = Nil(R)$.
(4) $Nil(R_{Nil(R)}) = \phi(Nil(R)) = Nil(\phi(R)) = Z(\phi(R))$.
(5) $T(\phi(R)) = R_{Nil(R)}$ is quasilocal with maximal ideal $Nil(\phi(R))$, and $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R))$ is the quotient field of $\phi(R)/Nil(\phi(R))$.

(6) If $R \in \mathcal{H}_0$ and $D = R/Nil(R)$, then $D' = R'/Nil(R)$ [2, Lemma 2.8].

The technique of idealization as in [24] is used in this paper to construct examples. Recall that for an R-module M, the idealization of M over R is the ring formed from $R \times M$ by defining addition and multiplication as $(r, a) + (s, m) = (r + s, a + m)$ and $(r, a)(s, m) = (rs, rm + sa)$, respectively.

2. ϕ-FC-Extensions

Let $R \subseteq S$ be a ring extension. Then $[R, S]$ denotes the set of all rings that are between R and S, and $(R : S) = \{r \in R \mid rS \subseteq R\}$ is the conductor of R in S. We start with the following (trivial) lemma.

Lemma 2.1. *Suppose that $R \subseteq S$ is a ring extension such that $Nil(R) = Nil(S)$. Then*

(1) $R/Nil(R) = S/Nil(R)$ *if and only if* $R = S$.
(2) $R \subseteq S$ *is an FC(FO)-extension if and only if* $R/Nil(R) \subseteq S/Nil(R)$ *is an FC(FO)-Extension.*
(3) $[R, S]$ *satisfies the d.c.c(a.c.c)-condition if and only if* $[R/Nil(R), S/Nil(R)]$ *satisfies the d.c.c(a.c.c)-condition.*
(4) $(R/Nil(R) : S/Nil(R)) = (R : S)/Nil(R)$.

The following result is a generalization of [23, Theorem 5].

Theorem 2.2. *Let $R \in \mathcal{H}_0$. Then each $\alpha \in T(R)$ is the root of a polynomial in $R[X]$ with unit coefficient (i.e. one of the coefficients is a unit) if and only if the integral closure of R (in $T(R)$) is a Prüfer ring. In particular, an integrally closed ring $R \in \mathcal{H}_0$ is a Prüfer ring if and only if each $\alpha \in T(R)$ is the root of a polynomial in $R[X]$ with unit coefficient.*

Proof. Let $D = R/Nil(R)$. Suppose that R' is a Prüfer ring. Let $\alpha \in T(R)$. Since D is a Prüfer domain by [3, Theorem 2.6] and $T(D) = T(R)/Nil(R)$, $\alpha + Nil(R)$ is the root of a polynomial in $D[X]$ with unit coefficient. Since an element $b \in R$ is a unit of R if and only if $b + Nil(R)$ is a unit of D, we conclude that α is the root of a polynomial in $R[X]$ with unit coefficient.

Conversely, suppose that each $\alpha \in T(R)$ is the root of a polynomial in $R[X]$ with unit coefficient. Then it is clear that each $\beta \in T(R)/Nil(R)$ is the root of a polynomial in $(R/Nil(R))[X]$ with unit coefficient. Since $T(R)/Nil(R)$ is the total quotient field of the integral domain $R/Nil(R)$, the integral closure of $R/Nil(R)$ (in $T(R)/Nil(R)$) is a Prüfer domain by [23, Theorem 5]. Since the integral closure of $R/Nil(R)$ is of the form of $R'/Nil(R)$ by [2, Lemma 2.8], we conclude that R' is a Prüfer ring by [3, Theorem 2.6]. $\square$

The following result is a generalization of [22, Corollary 1.2].

Corollary 2.3. *Let $R \in \mathcal{H}_0$. If d.c.c is satisfied in $[R, T(R)]$, then R' is a Prüfer ring. In particular, the integral closure of an FC-ring in $\mathcal{H}_0$ is a Prüfer ring.*

Proof. Since $[R, T(R)]$ satisfies the d.c.c, each $\alpha \in T(R)$ is the root of a polynomial in $R[X]$ with unit coefficient by [22, Proposition 1.1]. Thus the claim is now clear by Theorem 2.2 and by the fact that an FC-ring satisfies the d.c.c condition. □

Let S be a ring extension of a ring R. Then recall that S is said to be *strongly affine over* R if every subring B of S such that $R \subseteq B \subseteq S$ is finitely generated as a ring extension of R. The following result is a generalization of [22, Proposition 1.3].

Proposition 2.4. *Let $R \in \mathcal{H}_0$. If R is an FC-ring, then $T(R)$ is strongly affine over R; hence the integral closure of R (inside $T(R)$) is a finite R-module.*

Proof. Suppose that R is an FC-ring. Let $D = R/Nil(R)$. Since $T(D) = T(R)/Nil(R)$, D is an FC-domain by Lemma 2.1. Thus $T(D)$ is strongly affine over D by [22, Proposition 1.3]. It is easily verified that $T(D)$ is strongly affine over D if and only if $T(R)$ is strongly affine over R. Since $D' = R'/Nil(R)$ and D' is a finite D-module by [22, Proposition 1.3], it is easily verified that R' is a finite R-module. □

The following result is a generalization of [22, Theorem 1.5].

Theorem 2.5. *Let $R \in \mathcal{H}_0$ be an integrally closed ring. The following conditions are equivalent:*

(1) *R is a Prüfer ring with finitely many prime ideals;*
(2) *$R/Nil(R)$ is a Prüfer domain with finitely many prime ideals;*
(3) *R is a finite dimensional Prüfer ring with finitely many maximal ideals;*
(4) *$R/Nil(R)$ is a finite dimensional Prüfer domain with finitely many maximal ideals;*
(5) *$R/Nil(R)$ is an FC-domain;*
(6) *$R/Nil(R)$ is an FO-domain;*
(7) *R is an FO-ring;*
(8) *R is an FC-ring.*

Proof. Let $D = R/Nil(R)$. Then D is an integral domain with quotient field $T(R)/Nil(R)$. Since $D' = R'/Nil(R)$ and R is an integrally closed ring, we conclude that D is an integrally closed domain. We will prove

(2) $\Rightarrow$ **(3)** and **(8)** $\Rightarrow$ **(1)**. The reader should be able to verify the other implications. **(2)** $\Rightarrow$ **(3)**. Since D is a Prüfer domain with finitely many prime ideals, D is a finite dimensional Prüfer domain with finitely many maximal ideals by [22, Theorem 1.5]. Thus R is a finite dimensional Prüfer ring with finitely many maximal ideals by [3, Theorem 2.6]. **(8)** $\Rightarrow$ **(1)**. Since D is an FC-domain, D is a Prüfer domain with finitely many prime ideals by [22, Theorem 1.5]. Hence R is a Prüfer ring by [3, Theorem 2.6] and it is clear that R has finitely many prime ideals. □

Observe that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}_0$. Hence in view of Theorem 2.2, Corollary 2.3, and Proposition 2.4, we have the following corollary.

Corollary 2.6. *Let $R \in \mathcal{H}$. Then all the following statements hold:*

(1) *Each $\alpha \in R_{Nil(R)}$ is the root of a polynomial in $\phi(R)[X]$ with unit coefficient if and only if the integral closure of $\phi(R)$ (in $R_{Nil(R)}$) is a Prüfer ring. In particular, a ϕ- integrally closed ring $R \in \mathcal{H}$ is a ϕ-Prüfer ring if and only if each $\alpha \in T(R)$ is the root of a polynomial in $\phi(R)[X]$ with unit coefficient.*

(2) *If d.c.c is satisfied in $[\phi(R), R_{Nil(R)}]$, then $\phi(R)'$ is a Prüfer ring. In particular, the ϕ-integral closure of a $\phi - FC$-ring in $\mathcal{H}$ is a Prüfer ring.*

(3) *If R is a $\phi - FC$-ring, then $R_{Nil(R)}$ is strongly affine over $\phi(R)$; hence the integral closure of $\phi(R)$ (inside $R_{Nil(R)}$) is a finite $\phi(R)$-module.*

Theorem 2.7. *Let $R \in \mathcal{H}$. The following statements hold :*

(1) *R is a ϕ-FC-ring if and only if $R/Nil(R)$ is an FC-domain.*

(2) *R is a ϕ-FO-ring if and only if $R/Nil(R)$ is an FO-domain.*

PROOF. **(1)** Suppose that R is a ϕ-FC-ring. Then $\phi(R)$ is an FC-ring. Let $D = \phi(R)/Nil(\phi(R))$. Since $T(D) = T(\phi(R))/Nil(\phi(R)) = R_{Nil(R)}/Nil(\phi(R))$, we conclude that D is an FC-domain by Lemma 2.1. Since D is ring-isomorphic to $R/Nil(R)$ by [3, Lemma 2.5], we conclude that $R/Nil(R)$ is an FC-domain. Conversely, suppose that $F = R/Nil(R)$ is an FC-domain. Again, by Lemma 2.1 $\phi(R)$ is an FC-ring, and thus R is a ϕ-FC-ring.

(2) Just use a similar argument as in (1). □

Let $R \in \mathcal{H}$. Then R is a ϕ-Prüfer ring if and only if $R/Nil(R)$ is a Prüfer domain by [3, Theorem 2.6]. In view of Theorem 2.5, for a ring $R \in \mathcal{H}$ we have the following implications:

R is a ϕ-Prufer ring with finitely many prime ideals $\Leftrightarrow$ R is a ϕ-FC and a ϕ-integrally closed ring $\Leftrightarrow$ R is a ϕ-FO and a ϕ-integrally closed ring.

The following result is a generalization of [22, Corollary 1.6].

Corollary 2.8. *A ϕ-FC-ring in $\mathcal{H}$ has finitely many prime ideals.*

PROOF. Let $D = R/Nil(R)$. Since D is an FC-domain by Theorem 2.7, D has finitely many prime ideals by [22, Corollary 1.6], and hence it is clear that R has finitely many prime ideals. □

The following is an example of a non-domain FC-ring $R \in \mathcal{H}_0$ that is not an FO-ring.

Example 2.9. *Let J be the FC-domain that is not an FO-domain constructed in* [22, Example 1.7] *and let L be the quotient field of J. Set $R = J(+)L$. It is easily verified that $Z(R) = Nil(R) = \{0\}(+)L$ is a divided prime ideal of R, and hence $R \in \mathcal{H}_0$. Since $R/Nil(R)$ is ring-isomorphic to J, we conclude that $R/Nil(R)$ is an FC-domain that is not an FO-domain. Hence R is an FC-ring that is not an FO-ring by Lemma 2.1.*

The following result is a generalization of [22, Theorem 2.3].

Theorem 2.10. *Let $R \in \mathcal{H}_0$. Then R is an FC-ring if and only if a.c.c. and d.c.c. hold in both $[R, R']$ and $[R', T(R)]$.*

PROOF. Let $D = R/Nil(R)$. Then D is an integral domain with quotient field $T(R)/Nil(R)$ and $D' = R'/Nil(R)$. Suppose that R is an FC-ring. Then D is an FC-domain by Lemma 2.1. Thus a.c.c. and d.c.c. hold in both $[D, D']$ and $[D', T(D)]$ by [22, Theorem 2.3], and hence a.c.c. and d.c.c. hold in both $[R, R']$ and $[R', T(R)]$ by Lemma 2.1. Conversely, suppose that a.c.c. and d.c.c. hold in both $[R, R']$ and $[R', T(R)]$. Then a.c.c. and d.c.c. hold in both $[D, D']$ and $[D', T(D)]$ by Lemma 2.1. Thus D is an FC-domain by [22, Theorem 2.3]. Hence R is an FC-ring by Lemma 2.1. □

In view of Theorems 2.7, 2.10, and [22, Theorem 2.4], we have the following corollary.

Corollary 2.11. *Let $R \in \mathcal{H}$. The following statements are equivalent:*

(1) *R is a ϕ- FC-ring;*
(2) *a.c.c and d.c.c hold in both $[R/Nil(R), (R/Nil(R))']$ and $[(R/Nil(R))', R_{Nil(R)}/Nil(R_{Nil(R)})]$.*

The following result is a generalization of [22, Theorem 2.3].

Theorem 2.12. *Suppose that $R \in \mathcal{H}$ has finitely many maximal ideals. Then R is a ϕ-FC-ring if and only if R_M is a ϕ-FC-ring for each maximal ideal M of R.*

Proof. Set $D = R/Nil(R)$. Suppose that R is a ϕ-FC-ring. Let M be a maximal ideal of R. Since D is an FC-domain by Theorem 2.7, $D_{M/Nil(R)} = R_M/Nil(R_M)$ is an FC-domain by [22, Theorem 2.4]. Hence R_M is a ϕ-FC-ring by Theorem 2.7. Conversely, suppose that R_M is a ϕ-FC-ring for each maximal ideal M of R. Hence $R_M/Nil(R_M) = D_{M/Nil(R)}$ is an FC-domain by Theorem 2.7 for each maximal ideal M of R. Thus, $D = R/Nil(R)$ is an FC-domain by [22, Theorem 2.4], and hence R is a ϕ-FC ring by Theorem 2.7. □

Corollary 2.13. *Suppose that $R \in \mathcal{H}_0$ has finitely many maximal ideals. Then R is an FC-ring if and only if R_M is an FC-ring for each maximal ideal M of R.*

The following result is a generalization of [22, Theorem 2.14].

Theorem 2.14. *Let $R \in \mathcal{H}_0$ and let C be the conductor of R in R'. Then R is an FC-ring if and only if the following three conditions are satisfied:*

(1) *R' is a Prüfer ring with finitely many prime ideals.*
(2) *R' is a finite R-module.*
(3) *R/C is an Artinian ring.*

Proof. Let $D = R/Nil(R)$. Suppose that R is an FC-ring. Then the conditions (1) and (2) hold by Theorem 2.5, Corollary 2.8, and Proposition 2.4. Let J be the conductor of D in D'. Then $J = C/Nil(R)$ by Lemma 2.1. Since D is an FC-domain by Lemma 2.1 and $R/C \cong \frac{R/Nil(R)}{C/Nil(R)} \cong D/J$, we conclude that D/J is an Artinian ring by [22, Theorem 2.14], and hence R/C is an Artinian ring. Conversely, suppose that the conditions (1), (2), and (3) hold. Since $J = C/Nil(R)$ is the conductor of D in D' and $R/C \cong D/J$, D/J is an Artinian ring. Since R' is a finite R-module and $D' = R'/Nil(R)$, we conclude that D' is a finite D-module. Since R' is a Prüfer ring with finitely many prime ideals, D is a Prüfer domain with finitely many prime ideals by [3, Theorem 2.6]. Thus D is an FC-domain by [22, Theorem 2.14]. Hence R is an FC-ring by Lemma 2.1. □

In view of Theorem 2.14 and Theorem 2.7, we have the following corollary.

Corollary 2.15. *Let $R \in \mathcal{H}$, $D = R/Nil(R)$, and let C be the conductor of $\phi(R)$ in $\phi(R)'$. The following statements are equivalent:*

(1) R *is a* ϕ*-FC-ring.*
(2) *The following three conditions are satisfied:*
   (a) D' *is a Prüfer ring with finitely many prime ideals.*
   (b) D' *is a finite* D*-module.*
   (c) D/N *is an Artinian ring, where* N *is the conductor of* D *in* D'.

Combining Theorems 2.10, Corollary 2.13, and Theorem 2.14 we arrive at the following corollary.

Corollary 2.16. *Let* $R \in \mathcal{H}_0$*, and let* C *be the conductor of* R *in* R'*. The following statements are equivalent:*

(1) R *is an* FC*-ring.*
(2) *a.c.c. and d.c.c. hold in both* $[R, R']$ *and* $[R', T(R)]$.
(3) $Max(R)$ *is finite and* R_M *is an* FC*-ring for each maximal ideal* M *of* R.
(4) *The following three conditions are satisfied:*
   (a) R' *is a Prüfer ring with finite spectrum;*
   (b) R' *is finite* R*-module;*
   (c) R/C *is an Artinian ring.*

The following result is a generalization of [26, Corollary 3.4].

Theorem 2.17. *Let* $R \in \mathcal{H}_0$ *be a Prüfer ring. If* R *is an* FC*-ring, then each maximal chain* $R = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = T(R)$ *of overrings of* R *has length* $n =\mid Spec(R) \mid -1$.

Proof. Let $D = R/Nil(R)$. Then D is a Prüfer domain by [3, Theorem 2.6]. Let $R = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = T(R)$ be a maximal chain of overrings of R. Since $T(D) = T(R)/Nil(R)$, $D = R/Nil(R) \subset R_1/Nil(R) \subset R_2/Nil(R) \cdots \subset R_n/Nil(R) = T(D)$ is a maximal chain of overrings of D, and hence it has length $\mid Spec(D) \mid -1$ by [26, Corollary 3.4]. Since $\mid Spec(D) \mid = \mid Spec(R) \mid$, we conclude that the maximal chain $R = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = T(R)$ of overrings of R has length $\mid Spec(R) \mid -1$.

□

Corollary 2.18. *Let* $R \in \mathcal{H}$ *be a* ϕ*-Prüfer ring. If* R *is a* ϕ*-FC-ring, then the following statements hold:*

(1) *Each maximal chain* $\phi(R) = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = R_{Nil(R)}$ *of overrings of* $\phi(R)$ *has length* $n =\mid Spec(R) \mid -1$.
(2) *Each maximal chain* $R/Nil(R) = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = R_{Nil(R)}/Nil(R_{Nil(R)})$ *of overrings of* $R/Nil(R)$ *has length*

$n = \mid Spec(R) \mid -1$.

PROOF. Just observe that $\phi(R) \in \mathcal{H}_0$ and $\mid Spec(R) \mid = \mid Spec(\phi(R)) \mid = \mid Spec(R/Nil(R)) \mid$ by [16, Lemma 2.1]. □

The following result is a generalization of [17, Theorem 3.6 and Proposition 3.8].

Theorem 2.19. *Let $R \in \mathcal{H}$ be of finite Krull dimension $d \geq 1$. The following statements are equivalent:*

(1) *R is a ϕ-chained ring;*
(2) *$R/Nil(R)$ is a valuation domain;*
(3) $\mid [R/Nil(R), R_{Nil(R)}/Nil(R_{Nil(R)})] \mid = d+1$*;*
(4) $\mid [\phi(R), R_{Nil(R)}] \mid = d+1$*;*
(5) *For each chain of overrings $\phi(R) = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = R_{Nil(R)}$ of $\phi(R)$, we have $n \leq d$;*
(6) *For each chain of overrings $R/Nil(R) = R_0 \subset R_1 \subset R_2 \cdots \subset R_n = R_{Nil(R)}/Nil(R_{Nil(R)})$ of $R/Nil(R)$, we have $n \leq d$.*

PROOF. Let $D = R/Nil(R)$ and $F = \phi(R)/Nil(\phi(R))$. Then $T(D) \cong T(F) = R_{Nil(R)}/Nil(R_{Nil(R)})$. **(1)** $\Longleftrightarrow$ **(2)**. See [3, Lemma 2.7]. **(2)** $\Rightarrow$ **(3)**. Since D is ring-isomorphic to F by [3, Lemma 2.5], F is a valuation domain and the Krull dimension of F is d. Hence $\mid [F, T(F)] \mid = \mid [D, T(D)] \mid = d+1$ by [17, Theorem 3.6 and Proposition 3.8]. **(3)** $\Rightarrow$ **(4)** $\Rightarrow$ **(5)** $\Rightarrow$ **(6)** . These implications are clear since there is a one-to-one correspondence between the overrings of F and the overrings of $\phi(R)$. **(6)** $\Rightarrow$ **(1)**. By [17, Theorem 3.6 and Proposition 3.8], D is a valuation domain, and thus R is a ϕ-chained ring by [3, Lemma 2.7]. □

In the following result, we show that a ϕ-FC-ring is a pullback of an FC-domain.

Theorem 2.20. *Let $R \in \mathcal{H}$. Then R is a ϕ-FC-ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is an FC-subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

PROOF. Suppose $\phi(R)$ is ring-isomorphic to a ring A obtained from the given diagram. Then $A \in \mathcal{H}$ and $Nil(A) = Z(A) = M$. Since A/M is an FC-domain, A is a ϕ-FC-ring by Theorem 2.7(1), and thus R is a ϕ-FC-ring.

Conversely, suppose that R is a ϕ-FC-ring. Then, letting $T = R_{Nil(R)}$, $M = Nil(R_{Nil(R)})$, and $A = \phi(R)$ yields the desired pullback diagram. □

It is clear that if $R \in \mathcal{H}$ is a $\phi - FC$-ring, then R is an FC-ring. The following is an example of an FC-ring $R \in \mathcal{H}$ but R is not a $\phi - FC$-ring.

Example 2.21. *Let D be a Prüfer domain with infinitely many maximal ideals and let K be the quotient field of D. Set $R = D(+)(K/D)$. It is easily verified that $R \in \mathcal{H}$ and every nonunit of R is a zero-divisor of R. Thus $R = T(R)$, so R is ϕ-integrally closed. Hence R is an FC-ring. Since $R/Nil(R)$ is ring-isomorphic to D, we conclude that $R/Nil(R)$ is not an FC-domain by Corollary* **??**, *and thus R is not a $\phi - FC$-ring by Theorem 2.7(1).*

3. ϕ-FO-Extension

The results in this section are parallel to those for FC-extension in the previous section and the proofs are similar too. Hence we will only state the results of this section without giving proofs.

The following result is a generalization of [22, Theorem 3.1], also see [1, Theorem 2.6].

Theorem 3.1. *Let $R \in \mathcal{H}_0$. Then R is an FO-ring if and only if each of the sets $[R, R']$, and $[R', T(R)]$ is finite.*

The following result is a generalization of [22, Theorem 3.2].

Theorem 3.2. *Let $R \in \mathcal{H}_0$ with finitely many maximal ideals. Then R is an FO-ring if and only if R_M is an FO-ring for each maximal ideal M of R.*

Anderson, Dobbs, and Mullins [1] and [2] investigated finiteness of $[R, S]$ for a ring extension $R \subseteq S$. If $[R, S]$ is finite, they say $R \subseteq S$ has FIP. The following result is a generalization of [22, Theorem 3.4]

Theorem 3.3. *Let $R \in \mathcal{H}_0$, and let C be the conductor of R in R'. Then R is an FO-ring if and only if R' is a Prüfer ring with finitely many prime ideals and the extension $R/C \subset R'/C$ has FIP.*

Combining Theorem 3.1, 3.2, and 3.3 we arrive at the following corollary.

Corollary 3.4. *Let $R \in \mathcal{H}_0$, and let C be the conductor of R in R'. The following statements are equivalent:*

(1) *R is an FO-ring;*

(2) *R has finitely many maximal ideals and R_M is an FO-ring for each maximal ideal M of R;*

(3) *R' is a Prüfer ring with finitely many prime ideals and $R/C \subset R'/C$ has FIP.*

A similar argument as in Theorem 2.20, one can easily verify the following result.

Corollary 3.5. *Let $R \in \mathcal{H}$. Then R is a ϕ-FO-ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram:*

$$\begin{array}{ccc} A & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is an FO-subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.

Acknowledgment

The authors would like to thank the referee for his (her) valuable comments and suggestions.

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Received June 13, 2006

Revised version received September, 4, 2006

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